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BOUNDS ON THE BASE OF SOME PRIMITIVE NON-POWERFUL ZERO-SYMMETRIC SIGN PATTERN MATRICES

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Abstract

In 1994 Li et al. [3] extended the concept of the index of convergence from nonnegative matrices to powerful sign pattern matrices. In 2006, Liu and You extended the concept of the base from sign pattern matrices to non-powerful (and generalized) sign pattern matrices. In this paper, we study some primitive non-powerful zero-symmetric sign pattern matrix A, whose associated graph $D(A)$ is $v_{1k_1}; v_{21}, v_{2k_2}; k_1, k_2; l$)- lollipop or $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; \cdots; v_{m1}, v_{mk_m}; k_1, k_2, \cdots, k_m; l$)- lollipop. The bounds on the base of primitive non-powerful zero-symmetric sign pattern matrix A are obtained.

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1. Introduction

A sign pattern matrix (simple sign pattern or pattern) is the matrix, whose entries come from the set $\{+, -, 0\}$. The set of all $n \times n$ sign pattern is denoted by Q_n. Let $A = (a_{ij}) \in Q_n$, the qualitative class of A is defined by

$$Q(A) = \{B \in M_n(R) \mid \mathrm{sgn}(B) = A\}.$$

For a square sign pattern matrix A, notice that in the computations of (the signs of) the entries of the powers A^k, an "ambiguous sign", written as $\#$, may arise when we add a positive sign to a negative sign. For convenience, we call the set $\Gamma = \{+, -, 0, \#\}$ generalized sign set and define addition and multiplication involving the symbol $\#$ as follows (addition and multiplication which do not involve $\#$ are obvious):

$$(-) + (+) = (+) + (-) = \#, \quad \alpha + \# = \# + a = \# \text{ (for all } a \in \Gamma),$$

$$0 \cdot \# = \# \cdot 0 = 0, \quad b \cdot \# = \# \cdot b = \# \text{ (for all } b \in \Gamma \setminus \{0\}).$$

Matrices whose entries come from the set Γ are called *generalized sign pattern matrices*. Addition and multiplication of generalized sign pattern matrices are defined in the usual way, so that the sum and product (including powers) of generalized sign pattern matrices are still generalized sign pattern matrices. In this paper, we assume that all the matrix operations are operations of the matrices over the set Γ.

Let $A = (a_{ij})$ be a square generalized sign pattern matrix of order n. The associated generalized digraph $D(A)$ of A (possibly with loops) is defined to be the digraph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and arc set $E = \{(v_i, v_j) \mid a_{ij} \neq 0\}$. The associated generalized signed digraph $S(A)$ of A is obtained from $D(A)$ by assigning the sign of a_{ij} to each arc (i, j) in $D(A)$.

Let S be a signed digraph of order n. Then there is a sign pattern matrix A of order n, whose signed associated digraph $S(A)$ is S.

Thus there is a corresponding relation between a signed digraph of order n and a sign pattern matrix A of order n.

The graph-theoretical methods are often useful in the study of the powers of matrices, so we now introduce some graph-theoretical concepts.

A walk W in a digraph is a sequence of arcs: $e_1, e_2, \cdots, e_k$ such that the terminal vertex of e_i is the same as the initial vertex of e_{i+1} for $i = 1, 2, \cdots, k-1$. The number k of edges is called the *length of the walk* W, denoted by $l(W)$. The sign of the walk W (in a signed digraph), denoted by sgn$W$, is defined to be $\prod_{i=1}^{k} \operatorname{sgn}(e_i)$. Two walks W_1 and W_2 in a signed digraph is called a *pair* of *SSSD walks*, if they have the same initial vertex, same terminal vertex and same length, but they have different signs.

A square generalized sign pattern matrix A is called *powerful*, if each power of A contains no $\#$ entry. It is easy to see from the above relation between sign matrices and signed digraphs that a sign pattern A is powerful, if and only if the associated signed digraph $S(A)$ contains no pairs of *SSSD* walks.

Let A be a square generalized sign pattern matrix of order n and $A, A^2, A^3, \cdots$ be the sequence of powers of A. (Since there are only 4^{n^2} different generalized sign patterns of order n, there must be repetitions in the sequence.) Suppose A^l is the first power, that is repeated in the sequence. Namely, suppose l is the least positive integer such that there is a positive integer p such that

$$A^l = A^{l+p}, \tag{1.1}$$

then l is called the *generalized base* (or *simply base*) of A, and is denoted by $l(A)$. The least positive integer p such that (1.1) holds for $l = l(A)$ is called the *generalized period* (or *simply period*) of A, and is denoted by $p(A)$.

We say that S is powerful, if A is powerful, (i.e., S contains no pairs of *SSSD* walks). Also, we define $l(S) = l(A)$ and $p(S) = p(A)$.

As we know, a square matrix A of order n is reducible, if there exists a permutation matrix P of order n such that

$$PAP^T = \begin{pmatrix} B & 0 \\ D & C \end{pmatrix},$$

where B and C are square non-vacuous matrices. The matrix A is irreducible, if it is not reducible.

For a generalized sign pattern matrix A, we use $|A|$ to denote the $(0, +)$- matrix obtained from A by replacing each nonzero entry by $+$. Clearly, $|A|$ completely determines the zero pattern of A. We have $|AB| = \| A \| B \|$ for generalized sign pattern matrices A and B. In particular, we have

$$|A^k| = \| A |^k |.$$

A nonnegative square matrix B is primitive, if there exists a positive integer l such that $B^k > 0$ (A^k is entrywise positive). The least such k is called the *primitive exponent of* B, denoted by $\exp(B)$. A square generalized sign pattern matrix A is called *primitive*, if $|A|$ is primitive, and in this case, we define $\exp(A) = \exp(|A|)$. A digraph D is called a *primitive digraph*, if there is a positive integer k such that for each vertex x and vertex y (not necessarily distinct) in D, there exists a walk of length k from x to y. The least such k is called the *primitive exponent* of D, denoted by $\exp(D)$. As we know, a digraph D is primitive, if and only if D is strongly connected (or simply strong) and the greatest common divisor (or simply g.c.d.) of the lengths of all the cycles of D is 1.

It is well known that a square matrix A is irreducible, if and only if $D(A)$ is strong, and A is primitive, if and only if $D(A)$ is primitive, and in this case, we have $\exp(A) = \exp(D(A))$.

Also, a number of upper bounds for $\exp(D)$ can be established by using the Frobenius numbers defined as below.

Let $a_1, \cdots, a_k$ be positive integers. Define the Frobenius set $S(a_1, \cdots, a_k)$ as:

$$(a_1, \cdots, a_k) = \{r_1 a_1 + \cdots + r_k a_k \mid r_1, \cdots, r_k \text{ are nonnegative integers}\}.$$

It is well known that if $g.c.d.(a_1, \cdots, a_k) = 1$, then $S(a_1, \cdots, a_k)$ contains all the sufficiently large positive integers. In this case, we define the Frobenius number $\phi(a_1, \cdots, a_k)$ to be the least integer ϕ such that $m \in S(a_1, \cdots, a_k)$ for all integers $m \geq \phi$.

Clearly, $\phi(a_1, \cdots, a_k) - 1$ is not in $S(a_1, \cdots, a_k)$. It is also well known that if $g.c.d.(a, b) = 1$, then $\phi(a, b) = (a-1)(b-1)$.

A square generalized sign pattern matrix A is called *Zero-pattern-symmetric* (abbreviated *zero-symmetric* or simply ZS), if $|A|$ is symmetric. If matrix A is zero-symmetric, then $D(A)$ can be regarded as an undirected graph (possibly with loops). In this paper, we use an undirected graph (possibly with loops) as the associated digraph of a generalized zero-symmetric sign pattern matrix. We actually have both directions on each edge of the graph.

The graphs which we consider in this paper are undirected graphs. We actually have both directions on each edge of the graph. A path is a non-empty graph $P = (V, E)$ with vertex set $v_1, v_2, \cdots, v_k$ and edge set $E = \{v_1v_2, v_2v_3, \cdots, v_{k-1}v_k\}$. The number of edges of P is its length. P is also denoted by $v_1v_2 \cdots v_k$.

For an undirected walk W of graph G and two vertices x, y on W, let $Q_W(x \to y)$ be the shortest path from x to y on W. Let $Q(x \to y)$ be the shortest path from x to y on G. For a cycle C, if x and y are two (not necessarily distinct) vertices on C and P is a path from x to y along C, then $C \setminus P$ denotes the path or cycle from x to y along C obtained by deleting the edges of P.

Let $P_i = v_{i1}v_{i2} \cdots v_{ik_i}$ be a path of length $k_i - 1 (i = 1, 2, \cdots, m)$, and let C be a cycle of length l, where $v_{ik_i} \in V(C) (i = 1, 2, \cdots, m)$ and $v_{1k_1} =$

$v_{2k_2} = \cdots = v_{mk_m} = v$. Thus graph L' is obtained by the paths P_i and the cycle C; see Figure 1. P_i and C are denoted by $C(L')$ and $P_i(L')$, respectively. The connected graph L' is called a $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; \cdots; v_{m1}, v_{mk_m}; k_1, k_2, \cdots, k_m; l)$- lollipop.

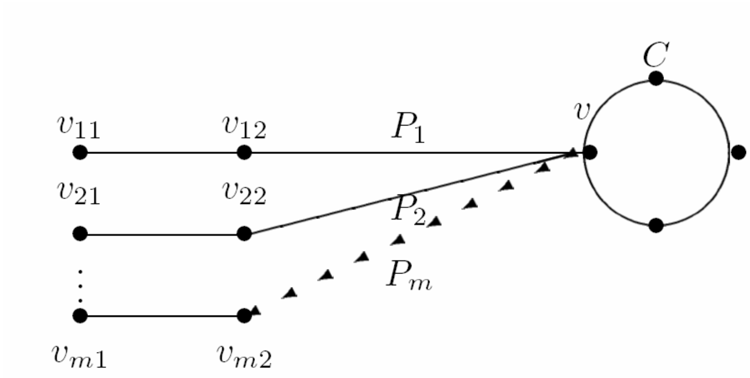

Figure1. Graph L', where $v_{1k_1} = v_{2k_2} = \cdots = v_{mk_m} = v$.

Let $P_i = v_{i1}v_{i2} \cdots v_{ik_i}$ be a path of length $k_i - 1$, where $i = 1, 2$ and let C be a cycle of length l, where $v_k, u_1 \in V(C)$. Thus graph L is obtained by the paths $P_i (i = 1, 2)$ and the cycle C; see Figure 2. P_i and C are denoted by $P_i(L)$ and $C(L)$, respectively. The connected graph L is called a $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; k_1, k_2; l)$- lollipop.

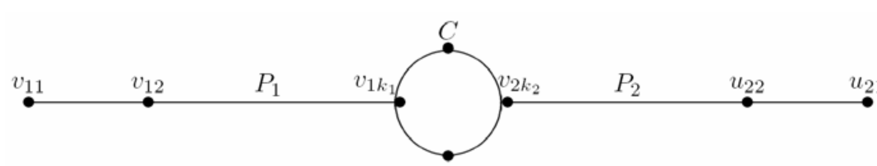

Figure 2. Graph L.

Obviously, Graph L. where $v_{1k_1} = v_{2k_2}$ is a special case of Graph L'. where $m = 2$.

In this paper, we study some primitive non-powerful zero-symmetric sign pattern matrix A. whose associated graph $D(A)$ is $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; k_1, k_2; l)$- lollipop or $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; \cdots; v_{m1}, v_{mk_m}; k_1, k_2, \cdots, k_m; l)$- lollipop. In Section 3, we consider the bounds on the base of $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; \cdots; v_{m1}, v_{mk_m}; k_1, k_2, \cdots, k_m; l)$- lollipop, then in Section 4, we consider the bounds on the base of $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; k_1, k_2; l)$-lollipop, where $v_{1k_1} \neq v_{2k_2}$. We obtain the bounds on the bases of primitive non-powerful zero-symmetric sign pattern matrix A.

2. Some Preliminaries

Theorem 2.1 [7]. *Let S be a primitive, non-powerful signed digraph. Then we have*

(1) *There is an integer k such that there exists a pair of SSSD walks of length k from each vertex x to each vertex y in S.*

(2) *If there exists a pair of SSSD walks of length k from each vertex x to each vertex y, then there also exists a pair of SSSD walks of length $k + 1$ from each vertex u to each vertex v in S.*

(3) *The minimal such k (as in (1)) is just $l(S)$- the base of S.*

A matrix with all entries equal to 1 is denoted by J. A matrix with all entries equal to $\#$ is denoted by $\#J$. Let A be a primitive generalized sign pattern matrix. Then

$$l(A) = \min\{k \mid A^k = J \text{ or } \#J\}. \tag{2.1}$$

Theorem 2.2 [7]. *If S is a primitive signed digraph, then S is non-powerful if and only if S contains a pair of cycles C_1 and C_2 (say, with lengths p_1 and p_2, respectively) satisfying one of the following two conditions:*

(A) *p_1 is odd and p_2 is even and $\operatorname{sgn} C_2 = -$;*

(B) *Both p_1 and p_2 are odd and $\operatorname{sgn} C_1 = -\operatorname{sgn} C_2$.*

A pair of cycles C_1 and C_2 satisfying (A) or (B) is a "distinguished cycle pair". It is easy to see that if C_1 and C_2 is a distinguished cycle pair with length p_1 and p_2, respectively, then the closed walks $W_1 = p_2C_1$ (walk around $C_1 p_2$ times) and $W_2 = p_1C_2$ have the same length p_1p_2 and the different signs:

$$(\operatorname{sgn} C_1)^{p_2} = -(\operatorname{sgn} C_2)^{p_1}. \tag{2.2}$$

For a primitive generalized sign pattern matrix A, the local base of A from i to j, denoted by $l_A(i, j)$, is the least integer k such that $(A^p)_{ij} = (A^k)_{ij}$ for all $p \geq k$. (Such that an integer k must exist by (2.1)). From this definition we have

$$l(A) = \max_{i, j \in V(D(A))} l_A(i, j). \tag{2.3}$$

Theorem 2.3 [2]. *For a primitive non-powerful sign pattern matrix A, suppose $R = \{l_1, \cdots, l_r\}$ is a set of cycle lengths in $D(A)$ with the property that g.c.d.$(l_1, \cdots, l_r) = 1$ and C', C'' are two cycles in $S(A)$ with the property that $(\operatorname{sgn} C')^{p_2} = -(\operatorname{sgn} C'')^{p_1}$, where p_1, p_2 are the lengths of C', C'', respectively. Let p be the least common multiple of p_1 and p_2, i.e., $p = $ l.c.m.(p_1, p_2), and $\phi_R = \phi(l_1, \cdots, l_r)$. Then*

(1) $l_A(i, j) \leq d_{R, C', C''}(i, j) + p + \phi_R$.

(2) $l(A) \leq \max_{i, j \in V(D(A))} d_{R, C', C''}(i, j) + p + \phi_R$.

Theorem 2.4 [**2**]. *Suppose A is a non-powerful sign pattern matrix of order n and $D(A)$ is a $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; \cdots; v_{m1}, v_{mk_m}; k_1, 2, \cdots, 2; 1)$- lollipop, where $n = k_1 + m - 1$ and $k_1 > 1$, then $l(A) = 2k_1$.*

Theorem 2.5 [**2**]. *Suppose A is a non-powerful sign pattern matrix of order n and $D(A)$ is a $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; \cdots; v_{m1}, v_{mk_m}; k_1, 1, \cdots, 1; 1)$- lollipop, where $n = k_1 > 1$, then $l(A) = 2n$.*

Theorem 2.6 [2]. *Suppose A is a sign pattern matrix of order n and $D(A)$ is a $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; \cdots; v_{m1}, v_{mk_m}; k_1, t, \cdots, t; l)$- lollipop, where $n = k_1 + l + m - 2$, $l > 1$, l is odd and $t \in \{1, 2\}$. If there exist no positive 2-cycles in $S(A)$, then*

(1) $l(A) = 2l + 2k_1 - 3$ *if* $k_1 \geq 2$;

(2) $l(A) = 2l + 1$ *if* $k_1 = 1$ *and* $m > 1$;

(3) $l(A) = 2l - 1$ *if* $k_1 = 1$ *and* $m = 1$.

3. Bounds on the Base of $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; \cdots; v_{m1}, v_{mk_m}; k_1, k_2, \cdots, k_m; 1)$- lollipop

Theorem 3.1. *Suppose A is a non-powerful sign pattern of order n and $D(A)$ is a $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; \cdots; v_{m1}, v_{mk_m}; k_1, k_2, \cdots, k_m; 1)$- lollipop, where $n = \sum_{i=1}^{m} k_i - (m-1)$, and $\max_{1 \leq i \leq m} k_i > 1$. Then*

$$l(A) = 2 \max_{1 \leq i \leq m} k_i.$$

Proof. It is clear that A is primitive. Let $K = \max_{1 \leq i \leq m} k_i$.

Suppose $l(A) = h$. Then there exists a pair of *SSSD* walks from v_{i1} to v_{i1} of length p for each integer $p \geq h$. Note that there is only one walk from v_{i1} to v_{i1} of length $2k_i - 1$, so $h > 2 \max_{1 \leq i \leq m} k_i - 1 = 2K - 1$. Then we only need to prove that $l(A) \leq 2 \max_{1 \leq i \leq m} k_i = 2K$.

Since A is a non-powerful sign pattern matrix, there exists a pair of cycles C', C'' satisfying that $(\operatorname{sgn} C')^{p_2} = -(\operatorname{sgn} C'')^{p_1}$ by (2.2), where p_1, p_2 are the lengths of C' and C'', respectively. Since $D(A)$ only has cycles with lengths of 1 and 2, it follows that $l.c.m.(p_1, p_2) = 2$. Suppose x and y are any two (not necessarily distinct) vertices in $S(A)$.

If $x, y \in \{v_{i1}, v_{i2}, \cdots, v_{ik_i}\}$, then set

$$W = \begin{cases} Q(x \to v_{i1}) + Q(v_{i1} \to v_{ik_i}) + Q(v_{ik_i} \to y) & \text{if } \; l(Q(x \to v_{i1}) \leq l(Q(y \to v_{i1}), \\ Q(x \to v_{ik_i}) + Q(v_{ik_i} \to v_{i1}) + Q(v_{i1} \to y) & \text{otherwise,} \end{cases}$$

and then $l(W) \leq 2(k_i - 1) \leq 2K - 2$. Otherwise let $x \in \{v_{i1}, v_{i2}, \ldots, v_{ik_i}\}$ and $y \in \{v_{j1}, v_{j2}, \ldots, v_{jk_j}\}$ where $i \neq j$, then set $W = Q(x \to v_{ik_i}) + Q(v_{jk_j} \to y)$, and thus $l(W) \leq (k_i - 1) + (k_j - 1) \leq 2K - 2$.

Take cycle-length set $R = \{1, 2\}$. Since in $D(A)$ any cycle must meet at least one vertex of $v_{i2}, v_{i3}, \cdots, v_{ik_i} (i = 1, 2, , \cdots, m)$, W must meet C', C''. Thus $d_{R, C', C''}(x, y) \leq l(W) \leq 2K - 2$. Then $l_A(x, y) \leq 2K - 2 + 2 + \phi_R = 2K = 2 \max_{1 \leq i \leq m} k_i$ by Theorem 2.3. Therefore, $l(A) \leq 2 \max_{1 \leq i \leq m} k_i$ by (2.3). So

$$l(A) = 2 \max_{1 \leq i \leq m} k_i. \qquad \square$$

Corollary 3.1. *Suppose A is a non-powerful sign pattern of order n and $D(A)$ is a $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; k_1, k_2; 1)$- lollipop, where $n = k_1 + k_2 - 1$, and $\max\{k_1, k_2 > 1$. Then*

$$l(A) = 2 \max\{k_1, k_2\}.$$

Theorem 3.2. *Suppose A is a sign pattern matrix of order n and $D(A)$ is a $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; \cdots; v_{m1}, v_{mk_m}; k_1, k_2, \cdots, k_m; l)$- lollipop, where $n = \sum_{i=1}^{m} k_i - (m - 1), l > 1$ and l is odd. If there exist no positive 2-cycles in $S(A)$, then*

(1) *$l(A) = 2l + 2k_1 - 3$ if $k_1 \geq 2$ and $k_2 = k_3 = \cdots = k_m = t \in \{1, 2\}$;*

(2) *$l(A) = 2l + 1$ if $k_1 = 1, k_2 = \cdots = k_m = 2$, where $m \geq 1$;*

(3) *$l(A) = 2l - 1$ if $k_i = 1$, where $i = 1, 2, \cdots, m$;*

(4) *$l(A) = 2l + 2 \max_{1 \leq i \leq m} k_i - 3$, if $k_1, k_2, \cdots, k_m \geq 2$.*

Proof. The results of (1)-(3) are obtained by Theorem 2.6. Therefore, we mainly prove (4).

Note that $D(A)$ is strongly connected and contains cycles of lengths 2 and l, where l is odd. Then A is primitive. Since there exist no positive 2-cycles in $S(A)$, (A) of Theorem 2.2 holds. So, A is non-powerful.

Consider $C(D(A))$, the cycle of the $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; \cdots; v_{m1}, v_{mk_m}; k_1, k_2, \cdots, k_m; l)$- lollipop. Let $C = C(D(A)) = v_1'v_2' \cdots v_l'v_1'$, where $v_1'v_2' \cdots v_l'$ are vertices of $C(D(A))$. Consider the directed cycles $C' = v_1' v_2' \cdots v_l'v_1'$ and $C'' = v_1'v_l'v_{l-1}' \cdots v_2'v_1'$. Note that there exist no positive 2-cycles in $S(A)$ and l is odd, so $\operatorname{sgn} C' = -\operatorname{sgn} C''$.

Let $K = \max\limits_{1 \le i \le m} k_i$.

Let x and y be any two (not necessarily distinct) vertices in $D(A)$. We discuss the following three cases.

Case 1. $x, y \in V(C)$.

If $l(Q_C(x \to y))$ is even, then set $W = Q_C(x \to y)$, otherwise set $W = C \setminus Q_C(x \to y)$. Then $l(W)$ is even and $l(W) \le l - 1 = 2K + l - 3$. Set $W_1 = W + C'$ and $W_2 = W + C''$. Thus the pair W_1, W_2 is a pair of *SSSD* walks from vertex x to y. Obviously $l(W_i)(i = 1, 2)$, is odd and $l(W_i) \le 2K + 2l - 3$, where $i = 1, 2$. Hence, there exists a pair of *SSSD* walks from x to y length $2K + 2l - 3$.

Case 2. $x, y \notin V(C)$.

If $x, y \in \{v_{i1}, v_{i2}, \cdots, v_{ik_i - 1}\}$, where $i \in \{1, 2, , \cdots, m\}$, then set

$$W = \begin{cases} Q(x \to v_{ik_i}) + Q(v_{ik_i} \to y) + C & \text{if } l(Q(x \to v_{ik_i})) + l(Q(v_{ik_i} \to y)) \text{ is odd,} \\ Q(x \to v_{ik_i}) + Q(v_{ik_i} \to y) & \text{otherwise,} \end{cases}$$

and then $l(W)$ is even and $l(W) \le 2(k_i - 1) + l - 1 \le 2K + l - 3$. Set $W_1 = W + C'$ and $W_2 = W + C''$. Thus the pair W_1, W_2 is a pair of *SSSD* walks from vertex x to y. Obviously $l(W_i)(i = 1, 2)$ is odd and $l(W_i) \leq 2K + 2l - 3$, where $i = 1, 2$. Hence, there exists a pair of *SSSD* walks from x to y length $2K + 2l - 3$.

Otherwise, let $x \in \{v_{i1}, v_{i2}, \cdots, v_{ik_i - 1}\}$ and $y \in \{v_{j1}, v_{j2}, \cdots, v_{jk_j - 1}\}$, where $i \neq j$, and set

$$W = \begin{cases} Q(x \to v_{ik_i}) + Q(v_{jk_j} \to y) + C & \text{if } l(Q(x \to v_{ik_i})) + l(Q(v_{jk_j} \to y)) \text{ is odd,} \\ Q(x \to v_{ik_i}) + Q(v_{jk_j} \to y) & \text{otherwise,} \end{cases}$$

and then $l(W)$ is even and $l(W) \leq (k_i - 1) + (k_j - 1) + l - 1 \leq 2K + l - 3$. Set $W_1 = W + C'$ and $W_2 = W + C''$. Thus the pair W_1, W_2 is a pair of *SSSD* walks from vertex x to y. Obviously $l(W_i)(i = 1, 2)$ is odd and $l(W_i) \leq 2K + 2l - 3$ where $i = 1, 2$. Hence there exists a pair of *SSSD* walks from x to y length $2K + 2l - 3$.

Case 3. Only one vertex of x, y belongs to $V(C)$.

Without loss of generality, we may assume $x \in V(C)$ and $y \in \{v_{j1}, v_{j2}, \cdots, v_{jk_j - 1}\}$. Set

$$W = \begin{cases} Q_C(x \to v_{jk_j}) + Q(v_{jk_j} \to y) & \text{if } l(Q_C(x \to v_{jk_j})) + l(Q(v_{jk_j} \to y)) \text{ is even,} \\ C \setminus Q_C(x \to v_{jk_j}) + Q(v_{jk_j} \to y) & \text{otherwise,} \end{cases}$$

and then $l(W)$ is even and $l(W) \leq (k_j - 1) + l - 1 \leq 2K + l - 3$ by the fact $k_j \geq 2$ for $j \in \{1, 2, \cdots, m\}$. Set $W_1 = W + C'$ and $W_2 = W + C''$. Thus the pair W_1, W_2 is a pair of *SSSD* walks from vertex x to y. Obviously $l(W_i)(i = 1, 2)$ is odd and $l(W_i) \leq 2K + 2l - 3$, where $i = 1, 2$. Hence, there exists a pair of *SSSD* walks from x to y length $2K + 2l - 3$.

Therefore

$$l(A) \leq 2K + 2l - 3 \leq 2 \max_{1 \leq i \leq m} k_i + 2l - 3. \tag{3.1}$$

Suppose the integer t satisfying $k_t = \max_{1 \leq i \leq m} k_i$.

We show that there is no pair of $SSSD$ walks of length $r = 2l + 2k_t - 4$ from vertex v_{t1} to v_{t1}. Suppose the pair W_1, W_2 is a pair of $SSSD$ walks of length r from v_{t1} to v_{t1}. Since W_i is the "union" of C_2 and C_l (a cycle of length l), $W_i = a_i C_2 + b_i C_l$, $a_i \geq 0$, $b_i \geq 0$ and b_i is even $(i = 1, 2)$.

If $b_1 = b_2 = 0$, then $a_1 = a_2$. Since there exist no positive 2-cycles in $S(A)$, $\operatorname{sgn}(W_1) = \operatorname{sgn}(W_2) = (-)^{a1}$. It contradicts that W_1 and W_2 have different signs. Therefore $b_1 = b_2 = 0$ does not hold. We may assume $b_1 > 0$. Note that b_1 is even, so $b_1 \geq 2$.

Since $b_1 \geq 2$, w_1 contains v_{tk_t}. Note that W_1 is a walk from v_{t1} to v_{t1}, so W_1 contains $v_{t1}, v_{t2}, \cdots, v_{tk_t}$. Then $l(W_1) \geq 2(k-1) + b_1 l \geq 2k_t - 2 + 2l$, which contradicts $l(W_1) = 2l + 2k_t - 4$. Hence, there exists no pair of $SSSD$ walks of length $r = 2l + 2k_t - 4$ from vertex v_{t1} to v_{t1}.

Therefore

$$l(A) \geq 2l + 2k_t - 3 = 2 \max_{1 \leq i \leq m} k_i + 2l - 3. \tag{3.2}$$

Combining (3.1) and (3.2), we have

$$l(A) = 2 \max_{1 \leq i \leq m} k_i + 2l - 3. \qquad \square$$

Theorem 3.3. *Suppose A is a non-powerful sign pattern matrix of order n and $D(A)$ is a $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; \cdots; v_{m1}, v_{mk_m}; k_1, k_2, \cdots, k_m; l)$- lollipop, where $n = \sum_{i=1}^{m} k_i - (m-1) + l$, $l > 1$ and l is odd. If there exist one vertex $w \in V(C)$ in $S(A)$ such that w is contained in a positive 2-cycle C' and a negative 2-cycle C'', then $l(A) \leq 2 \max_{1 \leq i \leq m} k_i + l - 1$.*

Proof. Note that $D(A)$ is strongly connected and contains cycles of lengths 2 and l, where l is odd. Then A is primitive. Let x and y be any two (not necessarily distinct) vertices in $S(A)$.

If $x, y \in V(C)$, then set $W = Q_C(x \to w) + Q_C(w \to y)$ and $l(W) \leq l - 1$, otherwise set $W = Q(x \to v) + 2Q_C(v \to w) + Q(w \to y)$ and $l(W) \leq (l-1) + 2\max_{1 \leq i \leq m}(k_i - 1) = 2\max_{1 \leq i \leq m} k_i + l - 3$.

Set $W_1 = W + C'$ and $W_2 = W + C''$. Thus the pair W_1, W_2 is a pair of *SSSD* walks from vertex x to y. Obviously $l(W_i) \leq 2\max_{1 \leq i \leq m} k_i + l - 1$, where $i = 1, 2$. Hence, there exists a pair of *SSSD* walks from x to y length $2\max_{1 \leq i \leq m} k_i + l - 1$. Therefore

$$l(A) \leq 2\max_{1 \leq i \leq m} k_i + l - 1. \qquad \square$$

Theorem 3.4. *Suppose A is a non-powerful sign pattern matrix of order n and $D(A)$ is a $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; \cdots; v_{m1}, v_{mk_m}; k_1, k_2, \cdots, k_m; l)$- lollipop, where $n = \sum_{i=1}^{m} k_i - (m-1) + l$, $l > 1$ and l is odd. If there exist one vertex $w \in \{v_{h1}, v_{h2}, \cdots, v_{hk_h - 1}\}$ for some $h \in \{1, 2, \cdots, m\}$ in $S(A)$ such that w is contained in a positive 2-cycle C' and a negative 2-cycle C'', then*

$$l(A) \leq \max_{1 \leq i \leq m, i \neq h} k_i + 2k_h + l - 4.$$

Proof. Note that $D(A)$ is strongly connected and contains cycles of length 2 and l, where l is odd. Then A is primitive. Let x and y be any two (not necessarily distinct) vertices in $S(A)$.

Let

$$T = \max_{1 \leq i \leq m, i \neq h} k_i.$$

If $x, y \in V(C)$, then set $W = Q_C(x \to v) + 2Q(v \to w) + Q_C(v \to y)$, thus $l(W) \leq l - 1 + 2(k_h - 2) = 2k_h + l - 5 \leq T + 2k_h + l - 6$. Otherwise, we consider the following cases.

Case 1. $x, y \notin V(C)$.

Subcase 1.1. $x, y \in \{v_{i1}, v_{i2}, \cdots, v_{ik_i - 1}\}$.

If $i = h$, then set $W = Q(x \to w) + Q(w \to y)$ and $l(W) \le 2(k_h - 2) = 2k_h - 4 \le T + 2k_h + l - 6$. Otherwise set $W = Q(x \to v) + 2Q(v \to w) + Q(v \to y)$ and $l(W) \le 2(k_i - 1) + 2(k_h - 2) = 2k_i + 2k_h - 6 \le T + 2k_h + l - 6$.

Subcase 1.2. One vertex of x and y is in $\{v_{h1}, v_{h2}, \cdots, v_{hk_h-1}\}$.

If $y \in \{v_{h1}, v_{h2}, \cdots, v_{hk_h-1}\}$, then set $W = Q(x \to v) + Q(v \to w) + Q(w \to y)$ and $l(W) \le T - 1 + k_h - 2 + k_h - 2 = T + 2k_h - 5 \le T + 2k_h + l - 6$. Otherwise set $W = Q(x \to w) + Q(w \to v) + Q(v \to y)$ and $l(W) \le T - 1 + k_h - 2 + k_h - 2 = T + 2k_h - 5 \le T + 2k_h + l - 6$.

Case 2. Only one vertex of x and y is in $V(C)$.

If $x \in V(C)$, then set

$$W = \begin{cases} Q_C(x \to v) + Q(v \to w) + Q(w \to y) & \text{if } y \in \{v_{h1}, v_{h2}, \cdots, v_{hk_h-1}\}, \\ Q_C(x \to v) + 2Q(v \to w) + Q(v \to y) & \text{otherwise,} \end{cases}$$

and

$$l(W) \le \begin{cases} \frac{l-1}{2} + k_h - 2 + k_h - 2 = 2k_h - 4 + \frac{l-1}{2} \le 2k_h + l - 5 & \text{if } y \in \{v_{h1}, v_{h2}, \cdots, v_{hk_h-1}\}, \\ \frac{l-1}{2} + 2(k_h - 2) + T - 1 \le 2k_h + T + l - 6 & \text{otherwise,} \end{cases}$$

$$\le 2k_h + T + l - 6.$$

Otherwise set

$$W = \begin{cases} Q(x \to w) + Q(w \to v) + Q_C(v \to y) & \text{if } x \in \{v_{h1}, v_{h2}, \cdots, v_{hk_h-1}\}, \\ Q(x \to v) + 2Q(v \to w) + Q_C(v \to y) & \text{otherwise,} \end{cases}$$

and

$$l(W) \le \begin{cases} \frac{l-1}{2} + k_h - 2 + k_h - 2 = 2k_h - 4 + \frac{l-1}{2} \le 2k_h + l - 5 & \text{if } x \in \{v_{h1}, v_{h2}, \cdots, v_{hk_h-1}\}, \\ \frac{l-1}{2} + 2(k_h - 2) + T - 1 \le 2k_h + T + l - 6 & \text{otherwise,} \end{cases}$$

$\leq 2k_h + T + l - 6 \leq T + 2k_h + l - 6.$

Set $W_1 = W + C'$ and $W_2 = W + C''$. Thus the pair W_1, W_2 is a pair of *SSSD* walks from vertex x to y. Obviously $l(W_i) \leq\leq T + 2k_h + l - 6$, where $i = 1, 2$. Hence, there exists a pair of *SSSD* walks from x to y length $\leq T + 2k_h + l - 6$. Therefore

$$l(A) \leq T + 2k_h + l - 6 = 2k_h + \max_{1 \leq i \leq m} k_i + l - 4. \qquad \square$$

4. Bounds on the Base of $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; k_1, k_2; l)$- lollipop, where $v_{1k_1} \neq v_{2k_2}$

Theorem 4.1. *Suppose A is a sign pattern matrix of order n and $D(A)$ is a $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; k_1, k_2; l)$- lollipop, where $n = k_1 + l + k_2 - 2$, $k_i \geq 2 (i = 1, 2)$ and l is odd. If there exist no positive 2-cycles in $S(A)$, then $l(A) \leq 2 \max\{k_1, k_2\} + \frac{3l - 5}{2}$.*

Proof. Note that $D(A)$ is strongly connected and contains cycles of lengths 2 and l, where l is odd. Then A is primitive. Since there exist no positive 2-cycles in $S(A)$, (A) of Theorem 2.2 holds. So A is non-powerful. Clearly $l \geq 3$.

Let x and y be any two (not necessarily distinct) vertices in $S(A)$.

If $x, y \in V(C)$, then set $W = Q_C(x \to y)$ and thus $l(W) \leq \frac{l-1}{2} \leq 2 \max\{k_1, k_2\} + \frac{l-5}{2}$. Otherwise, we consider the following cases.

Case 1. $x, y \notin V(C)$.

If $x, y \in \{v_{i1}, v_{i2}, \cdots, v_{ik_i - 1}\}$ for $i = 1, 2$, then set

$$W = \begin{cases} Q(x \to v_{i1}) + Q(v_{i1} \to v_{ik_i}) + Q(v_{ik_i} \to y) & \text{if } l(Q(x \to v_{i1})) \leq l(Q(y \to v_{i1})), \\ Q(x \to v_{ik_i}) + Q(v_{ik_i} \to v_{i1}) + Q(v_{i1} \to y) & \text{otherwise,} \end{cases}$$

and thus $l(W) \leq 2(k_i - 1) \leq 2\max\{k_1, k_2\} + \frac{l-5}{2}$.

Otherwise only one vertex of x, y is in $\{v_{11}, v_{12}, \cdots, v_{1k_1-1}\}$, and set

$$W = \begin{cases} Q(x \to v_{1k_1}) + Q_C(v_{1k_1} \to v_{2k_2}) + Q(v_{2k_2} \to y) & \text{if } x \in \{v_{11}, v_{12}, \cdots, v_{1k_1-1}\}, \\ Q(x \to v_{2k_2}) + Q_C(v_{2k_2} \to v_{1k_1}) + Q(v_{1k_1} \to y) & \text{otherwise,} \end{cases}$$

and thus $l(W) \leq \frac{l-1}{2} + k_1 - 1 + k_2 - 1 \leq 2\max\{k_1, k_2\} + \frac{l-5}{2}$.

Case 2. Only one vertex x, y is in $V(C)$.

Set

$$W = \begin{cases} Q(x \to v_{ik_i}) + Q_C(v_{ik_i} \to y) & \text{if } x \in \{v_{i1}, v_{i2}, \cdots, v_{ik_i-1}\} \text{ for } i = 1, 2, \\ Q_C(x \to v_{ik_i}) + Q(v_{ik_i} \to y) & \text{if } x \in \{v_{i1}, v_{i2}, \cdots, v_{ik_i-1}\} \text{ for } i = 1, 2, \end{cases}$$

and thus $l(W) \leq \frac{l-1}{2} + k_1 - 1 \leq 2\max\{k_1, k_2\} + \frac{l-5}{2}$.

Consider $C(D(A))$, the cycle of the $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; k_1, k_2; l)$-lollipop. Let $C = C(D(A)) = v_1'v_2' \cdots v_l'v_1'$, where $v_1'v_2' \cdots v_l'$ are vertices of $C(D(A))$. Consider the directed cycles $C' = v_1'v_2' \cdots v_l'v_1'$ and $C'' = v_1'v_l'v_{l-1}' \cdots v_2'v_1'$. Note that there exist no positive 2-cycles in $S(A)$ and l is odd, so $\operatorname{sgn} C' = -\operatorname{sgn} C''$.

Set $W_1 = W + C'$ and $W_2 = W + C''$. Thus the pair W_1, W_2 is a pair of *SSSD* walks from x to y. We see that $l(W_i) = l(W) + l \leq 2\max\{k_1, k_2\} + \frac{3l-5}{2}$, where $i = 1, 2$. Hence, there exists a pair of *SSSD* walks from x to y of length $2\max\{k_1, k_2\} + \frac{3l-5}{2}$.

Therefore $l(A) \leq 2\max\{k_1, k_2\} + \frac{3l-5}{2}$. □

Theorem 4.2. *Suppose A is a sign pattern matrix of order n and $D(A)$ is a $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; k_1, k_2; l)$- lollipop, where $n = k_1 + l + k_2 - 2$, $k_i \geq 2(i = 1, 2)$, l is odd and $l \not\equiv 3(\mathrm{mod}\ 4)$. If there exist no positive 2-cycles in $S(A)$, then $l(A) = 2\max\{k_1, k_2\} + \frac{3l-5}{2}$.*

Proof. It is clear that by Theorem 4.1,

$$l(A) \leq 2\max\{k_1, k_2\} + \frac{3l-5}{2}. \tag{4.1}$$

Without loss of generality, assume that $k_1 = \max\{k_1, k_2\}$. Then we show that there is no pair of *SSSD* walks of length $r = 2k_1 + \frac{3l-5}{2} - 1$ from vertex v_{11} to v_{11}. Let the pair W_1, W_2 be a pair of *SSSD* walks of length r from v_{11} to v_{11}. Since W_i is the "union" of C_2 and C_l (a cycle of length l), $W_i = a_i C_2 + b_i C_l$, $a_i \geq 0$, $b_i \geq 0$.

If $b_i \geq 2(i = 1, 2)$, then $l(W_i) \geq 2(k_i - 1) + b_i l \geq 2k_i - 2 + 2l$, which contradicts $l(W_i) = 2k_1 + \frac{3l-5}{2} - 1$. Therefore, $b_i \leq 1$ for each $i \in \{1, 2\}$.

If $b_1 = b_2 = 0$, then $a_1 = a_2$. Since there exist no positive 2-cycles in $S(A)$, $\mathrm{sgn}(W_1) = \mathrm{sgn}(W_2) = (-)^{a_1}$. It contradicts that W_1 and W_2 have different signs. Therefore, $b_1 = b_2 = 0$ does not hold.

If $(b_1 = 1, b_2 = 0)$ or $(b_1 = 0, b_2 = 1)$, then $l = 2|a_1 - a_2|$, which contradicts that l is odd.

If $b_1 = b_2 = 1$, then $a_1 = a_2 = \frac{4k_1 + l - 7}{4} = k_1 + \frac{l-3}{4} - 1$, which contradicts that a_1 and a_2 are integers for $l \not\equiv 3(\mathrm{mod}\ 4)$.

Therefore, there is no pair of *SSSD* walks of length $r = 2k_1 + \frac{3l-5}{2} - 1$ from vertex v_{11} to v_{11}. Thus

$$l(A) \geq 2k_1 + \frac{3l-5}{2} = 2\max\{k_1, k_2\} + \frac{3l-5}{2}. \tag{4.2}$$

Combining (4.1) and (4.2), we have

$$l(A) = 2\max\{k_1, k_2\} + \frac{3l-5}{2}. \qquad \square$$

Theorem 4.3. *Suppose A is a non-powerful sign pattern matrix of order n and $D(A)$ is a $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; k_1, k_2; l)$- lollipop, where $n = k_1 + k_2 + l - 2$, $l > 1$, $k_1, k_2 \geq 2$ and l is odd. If there exist one vertex $w \in V(C)$ in $S(A)$ such that w is contained in a positive 2-cycle C' and a negative 2-cycle C'', then $l(A) \leq 2\max\{k_1, k_2\} + l - 1$.*

Proof. Note that $D(A)$ is strongly connected and contains cycles of length 2 and l, where l is odd. Then A is primitive. Let x and y be any two (not necessarily distinct) vertices in $S(A)$.

If $x, y \in V(C)$, then set $W = Q_C(x \to w) + Q_C(w \to y)$ and $l(W) \leq l - 1 \leq 2\max\{k_1, k_2\} + l - 3$; If $x, y \in \{v_{i1}, v_{i2}, \cdots, v_{ik_i-1}\}$ for $i \in \{1, 2\}$, then set $W = Q(x \to v_{ik_i}) + 2Q_C(v_{ik_i} \to w) + Q(v_{ik_i} \to y)$, and then $l(W) \leq 2(k_i - 1) + l - 1 = 2k_i + l - 3 \leq 2\max\{k_1, k_2\} + l - 3$; If $x \in \{v_{i1}, v_{i2}, \cdots, v_{ik_i-1}\}$ and $y \in \{v_{j1}, v_{j2}, \cdots, v_{jk_j-1}\}$, where $i, j \in \{1, 2\}$ and $i \neq j$, set $Q(x \to v_{ik_i}) + Q_C(v_{ik_i} \to w) + Q_C(w \to v_{jk_j}) + Q(v_{jk_j} \to y)$, and then $l(W) \leq (k_1 - 1) + (k_2 - 1) + l - 1 = k_1 + k_2 + l - 3 \leq 2\max\{k_1, k_2\} + l - 3$; Otherwise, only one vertex of x, y belongs to $V(C)$, and for $i \in \{1, 2\}$, set

$$W = \begin{cases} Q_C(x \to w) + Q_C(w \to v_{ik_i}) + Q(v_{ik_i} \to y) & \text{if } x \in V(C), \\ Q(x \to v_{ik_i}) + Q_C(v_{ik_i} \to w) + Q_C(w \to y) & \text{if } y \in V(C), \end{cases}$$

and

$$l(W) \leq (k_i - 1) + l - 1 = k_i + l - 2 \leq 2\max\{k_1, k_2\} + l - 3.$$

Let $W_1 = W + C'$ and $W_2 = W + C''$. Thus there exists a pair of *SSSD* walks W_1 and W_2 from x to y. Hence, there exists a pair of *SSSD* walks from x to y of length $2\max\{k_1, k_2\} + l - 1$. Thus $l(A) \le 2\max\{k_1, k_2\} + l - 1$. □

Theorem 4.4. *Suppose A is a non-powerful sign pattern matrix of order n and $D(A)$ is a $(v_{11}, v_{1k_1}; v_{21}, v_{2k_2}; k_1, k_2; l)$- lollipop, where $n = k_1 + k_2 + l - 2$, $l > 1$, $k_1 \ge 3$, $k_2 \ge 2$ and l is odd. If there exist one* vertex *$w \in \{v_{11}, \cdots, v_{1k_1-1}\}$ in $S(A)$ such that w is contained in a positive 2-cycle C' and a negative 2-cycle C'', then*

$$l(A) \le 2k_1 + 2k_2 + l - 5.$$

Proof. Note that $D(A)$ is strongly connected and contains cycles of length 2 and l, where l is odd. Then A is primitive. Let x and y be any two (not necessarily distinct) vertices in $S(A)$.

If $x, y \in V(C)$, then set $W = Q_C(x \to v_{1k_1}) + 2Q(v_{1k_1} \to w) + Q_C(v_{1k_1} \to y)$, thus $l(W) \le l - 1 + 2(k_1 - 2) = 2k_1 + l - 5 \le 2k_1 + 2k_2 + l - 7$. Otherwise, we consider the following cases.

Case 1. $x, y \notin V(C)$.

Subcase 1.1. $x, y \in \{v_{i1}, v_{i2}, \cdots, v_{ik_i-1}\}$ for $i \in \{1, 2\}$.

If $i = 1$, then set $W = Q(x \to w) + Q(w \to y)$ and $l(W) \le 2(k_1 - 2) = 2k_1 - 4 \le 2k_1 + 2k_2 + l - 7$. Otherwise set $W = Q(x \to v_{2k_2}) + 2Q_C(v_{2k_2} \to v_{1k_1}) + 2Q(v_{1k_1} \to w) + Q(v_{2k_2} \to y)$ and $l(W) \le 2k_2 + 2k_1 + l - 7$.

Subcase 1.2. One vertex of x and y is in $\{v_{21}, v_{22}, \cdots, v_{2k_2-1}\}$.

If $y \in \{v_{11}, v_{12}, \cdots, v_{1k_1-1}\}$, then set $W = Q(x \to v_{2k_2}) + Q_C(v_{2k_2} \to v_{1k_1}) + Q(v_{1k_1} \to w) + Q(w \to y)$ and $l(W) \le (k_2 - 1) + \frac{l-1}{2} + (k_1 - 2) + (k_1 - 2) = k_2 + \frac{l-1}{2} + 2k_1 - 5 \le 2k_1 + 2k_2 + l - 7$. Otherwise set $W = Q(x \to w) + Q(w \to v_{1k_1}) + Q_C(v_{1k_1} \to v_{2k_2}) + Q(v_{2k_2} \to y)$ and $l(W) \le (k_1 - 2) + (k_1 - 2) + \frac{l-1}{2} + (k_2 - 1) = k_2 + \frac{l-1}{2} + 2k_1 - 5 \le 2k_1 + 2k_2 + l - 7$.

Case 2. Only one vertex of x and y is in $V(C)$.

Subcase 2.1. $x \in \{v_{i1}, v_{i2}, \cdots, v_{ik_i-1}\}$.

If $i \neq 1$, then set $W = Q(x \to v_{2k_2}) + Q_C(v_{2k_2} \to v_{1k_1}) + 2Q(v_{1k_1} \to w) + Q_C(v_{1v_1} \to y)$ and $l(W) \leq (k_2 - 1) + 2(k_1 - 2) + (l - 1) = k_2 + 2k_1 + l - 6 \leq 2k_1 + 2k_2 + l - 7$. Otherwise set $W = Q(x \to w) + Q(w \to v_{1k_1}) + Q_C(v_{1v_1} \to y)$ and $l(W) \leq (k_1 - 2) + (k_1 - 2) + \frac{l-1}{2} = \frac{l-1}{2} + 2k_1 - 4 \leq 2k_1 + 2k_2 + l - 7$.

Subcase 2.2. $y \in \{v_{i1}, v_{i2}, \cdots, v_{ik_i-1}\}$.

If $i \neq 1$, then set $W = Q_C(x \to v_{1k_1}) + 2Q(v_{1k_1} \to w) + Q_C(v_{1k_1} \to v_{2k_2}) + Q(v_{2k_2} \to y)$ and $l(W) \leq (k_2 - 1) + 2(k_1 - 2) + (l - 1) = k_2 + 2k_1 + l - 6 \leq 2k_1 + 2k_2 + l - 7$. Otherwise set $W = Q_C(x \to v_{1k_1}) + Q(v_{1k_1} \to w) + Q(w \to y)$ and $l(W) \leq (k_1 - 2) + (k_1 - 2) + \frac{l-1}{2} = \frac{l-1}{2} + 2k_1 - 4 \leq 2k_1 + 2k_2 + l - 7$.

Set $W_1 = W + C'$ and $W_2 = W + C''$. Thus the pair W_1, W_2 is a pair of $SSSD$ walks from vertex x to y. Obviously $l(W_i) \leq 2k_1 + 2k_2 + l - 5$ where $i = 1, 2$. Hence, there exists a pair of $SSSD$ walks from x to y length $2k_1 + 2k_2 + l - 5$. Therefore,

$$l(A) \leq 2k_1 + 2k_2 + l - 5. \qquad \square$$

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